Journal of Statistics and Computer Science
Vol. 3, Issue 1, 2024, pp. 41-49
https://doi.org/10.47509/JSCS.2024.v03i01.03

Recursive Estimation in Restricted Exponential Autoregressive Models

N. Boumaila* and M. Merzougui*.
*LaPS Laboratory

Badji Mokhtar-Annaba University, 12, P.O.Box, 23000 Annaba, Algeria.
nadia.boumaila@univ-annaba.dz mouna.merzougui@univ-annaba.dz

Abstract

This study proposes a recursive estimation algorithm for the restricted exponential autoregressive ($EXPAR$) model. The recursive least squares (RLS) theory is based on the matrix inversion lemma. It is shown that the RLS estimators are asymptotically efficient. A short simulation study shows the high performance of the obtained estimators.

Key Words: Restricted exponential autoregressive model, On line estimation algorithm, RLS method, The matrix inversion lemma.

Mathematics Subject Classification 62F12; 62M10.

1. Introduction

The most standard estimate technique for getting a good estimator with a given sample size is the least squares approach (LS). Nevertheless, the need to treat progressive size has gotten more and more important. In fact, handling variable-sized series is necessary in many fields, such as signal processing, control, and instantaneous applications. An elegant approach to recursive identification is to derive it from off-line estimation, so the LS method can be carried out recursively. For more details, see Ljung and Söderström (1983). The current estimator is derived by incorporating new data into the previous estimator, which allows for continuous refinement and potential improvements in accuracy. The main tool for obtaining the RLS algorithm is the matrix inversion lemma, which leads to a formula without matrix inversion and offers excellent performance in terms of computation and memory space.
The RLS method was successfully applied to many models of time series; one of them is the $EXPAR$ model; see Xu et al. (2019) and Xu et al. (2020). This model was introduced

Article History
Received : 01 March 2024; Revised : 29 May 2024, Accepted : 05 June 2024; Published : 30 June 2024

To cite this paper
N. Boumaila & M. Merzougui (2024). Recursive Estimation in Restricted Exponential Autoregressive Models. *Journal of Statistics and Computer Science.* 3(1), 41-49.

by Ozaki (1980) and can exhibit nonlinear phenomena like limit cycles, jump phenomena, and non-normality. Due to the nonlinearity of the model, the obtained RLS algorithm was nonconventional, and nonlinear optimization was used. Precisely in the context of real-time estimation, Shi and Aoyama (1997) suggested a direct estimation from the data for the nonlinear parameter. Messaoud et al. (2006) utilized this method to model the vibrations and disturbances taking place during drilling. The Restricted $EXPAR$ model is therefore derived. It is worth noting that the restricted model is linear for the unknown parameters but still nonlinear with regard to the variable and retains all of its nonlinear behavior. The approach will perform well if the nonlinear parameter of the model is known from earlier research or can be readily determined in the context of real-time data.

In this paper, we present an online recursive least squares algorithm for the estimation of the restricted $EXPAR$ model. This approach is based on the matrix inversion lemma, which avoids matrix inversion and is good in terms of effective real-time computing and small memory requirements.

The structure of the paper is as follows: In Section 2, we review the definition and LS estimation for the restricted $EXPAR(p)$ model. We give the RLS estimators and prove their asymptotic properties under mild conditions in Section 3. Finally, a short simulation study is given in Section 4.

2. Restricted EXPAR(p) process

The restricted exponential autoregressive $EXPAR\,(p)$ process is given by the formula

$$Y_t = \sum_{j=1}^{p} \left(\theta_{1,j} + \theta_{2,j} \exp\left(-\gamma Y_{t-1}^2\right)\right) Y_{t-j} + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{1}$$

where $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is $i.i.d\,(0, \sigma^2)$. The slope parameter, $\gamma > 0$, is known. A heuristic determination of it from data

$$\widehat{\gamma} = -\frac{\log \epsilon}{\max\limits_t Y_t^2},$$

where ϵ is a small number.(cf. Shi et al. (2001)).

Let $\theta = (\theta_{1,1}, \theta_{2,1}, ..., \theta_{1,p}, \theta_{2,p})' \in \mathbb{R}^{2p}$, the vector of unknown parameters, and $\varphi\,(t) =$

$\left(Y_{t-1}, Y_{t-1}\exp\left(-\gamma Y_{t-1}^{2}\right), ..., Y_{t-p}, Y_{t-p}\exp\left(-\gamma Y_{t-1}^{2}\right)\right)'$. Equation (1) can be written in regression form

$$Y_t = \theta'\varphi\left(t\right) + \varepsilon_t, \quad t \in \mathbb{Z}, \tag{2}$$

We make the following assumptions:

$A1$: We suppose that the process is strictly stationary. If this assumption is violated, alternative methods such as using time-varying parameters or incorporating non-stationary components may be considered.

$A2$: The white noise $\{\varepsilon_t;\ t \in \mathbb{Z}\}$ is such that $E\left(\varepsilon_t^4\right) < \infty$, for any $t \in \mathbb{Z}$ which means that $E\left(Y_t^4\right) < \infty$. In case this assumption is violated, robust estimation techniques or methods requiring weaker moment conditions could be employed.

The least squares (LS) estimator minimizes the equation error with respect to θ

$$V_n\left(\theta\right) = \frac{1}{n}\sum_{t=1}^{n}\left(Y_t - \theta'\varphi\left(t\right)\right)^2,$$

where n is the size of the data. The criterion is quadratic in θ, thus

$$\widehat{\theta}_n = \left(\sum_{t=1}^{n}\varphi\left(t\right)\varphi\left(t\right)'\right)^{-1}\left(\sum_{t=1}^{n}\varphi\left(t\right)Y_t\right). \tag{3}$$

If θ is the true value, we can prove that the LS estimator $\widehat{\theta}_n$ converges asymptotically to θ and

$$\sqrt{n}\left(\widehat{\theta}_n - \theta\right) \xrightarrow{d} N\left(0, \sigma^2\Gamma^{-1}\right),$$

where $\Gamma = E\left(\varphi\left(t\right)\varphi\left(t\right)'\right)$ and $\xrightarrow{d}$ denotes convergence in distribution.

3. Recursive least squares estimation algorithm

Let $Y_1, ..., Y_t$ denote the available observations, until time t, of the restricted $EXPAR(p)$ model. The adopted optimality criterion in RLS estimation is the mean square error. Thus, the problem consists of finding the argument that minimizes

$$V_t\left(\theta\right) = \frac{1}{t}\sum_{k=1}^{t}\left(Y_k - \theta'\varphi\left(k\right)\right)^2. \tag{4}$$

The proposition below gives the RLS algorithm, which provides optimal estimators for the restricted $EXPAR$.

Proposition 1

The RLS algorithm for estimating the parameters of an $EXPAR(p)$ model is given by the following system of recurring equations:

$$\begin{cases} \widehat{\theta}_t = \widehat{\theta}_{t-1} + \dfrac{P_{t-1}\varphi(t)}{\varphi(t)' P_{t-1}\varphi(t)+1}\left(Y_t - \widehat{\theta}'_{t-1}\varphi(t)\right), \\ P_t = P_{t-1} - \dfrac{P_{t-1}\varphi(t)\varphi(t)' P_{t-1}}{\varphi(t)' P_{t-1}\varphi(t)+1}. \end{cases} \quad (5)$$

A standard choice of initial values is $P_0 = C.I$ and $\widehat{\theta}(0) = 0$, where C is some large constant, for example, $C = 10^6$, and I denotes the identity matrix.

Proof

From the objective function (4), the LS estimator $\widehat{\theta}_t$ is given by

$$\left(\sum_{k=1}^{t}\varphi(k)\varphi(k)'\right)\widehat{\theta}_t = \left(\sum_{k=1}^{t}\varphi(k)Y_k\right). \quad (6)$$

Denote

$$R_t = \sum_{k=1}^{t}\varphi(k)\varphi(k)',$$

it follows that

$$R_t = R_{t-1} + \varphi(t)\varphi(t)'. \quad (7)$$

After some straightforward modifications, one may derive the recursive equation using (6) and (7)

$$\widehat{\theta}_t = \widehat{\theta}_{t-1} + R_t^{-1}\varphi(t)\left(Y_t - \widehat{\theta}'_{t-1}\varphi(t)\right). \quad (8)$$

The algorithm (8) is not suited for computing because we must invert a $p \times p$ matrix in each step, so we introduce $P_t = R_t^{-1}$. The matrix inversion lemma can be used to update P_t directly

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B\left(DA^{-1}B + C^{-1}\right)^{-1}DA^{-1},$$

see, for example, Ljung and Söderström, 1983, Lemma 2.1, p. 19. We obtain

$$P_t = P_{t-1} - \frac{P_{t-1}\varphi(t)\varphi(t)' P_{t-1}}{\varphi(t)' P_{t-1}\varphi(t)+1}.$$

hence the algorithm (5) stating that the new estimate is equal to the previous estimate plus the prediction error multiplied by a gain.

The following proposition states the asymptotic properties of the RLS estimator.

Proposition 2

Under assumptions $A1$ and $A2$,

i) $\widehat{\theta}_t \to \theta$, almost surly for $t \to \infty$.

ii) $\sqrt{t}\left(\widehat{\theta}_t - \theta\right) \xrightarrow{d} N\left(0, \sigma^2\Gamma^{-1}\right).$

Proof

The RLS algorithm needs an initial value to start up, the estimates resulting from (8) are then

$$\widehat{\theta}_t = \left(P_0^{-1} + \sum_{k=1}^{t} \varphi(k)\varphi(k)'\right)^{-1} \left(P_0^{-1}\widehat{\theta}_0 + \sum_{k=1}^{t} \varphi(k) Y_k\right), \tag{9}$$

for $P(0)^{-1} \to 0$, the recursive estimates are asymptotically similar to

$$\widehat{\theta}_t = \left(\sum_{k=1}^{t} \varphi(k)\varphi(k)'\right)^{-1} \left(\sum_{k=1}^{t} \varphi(k) Y_k\right), \tag{10}$$

that is, the recursive estimate becomes closer to the off-line estimate and has the same asymptotic properties. Substituting Y_k by (2) we obtain

$$\widehat{\theta}_t = \theta + \left(\frac{1}{t}\sum_{k=1}^{t} \varphi(k)\varphi(k)'\right)^{-1} \left(\frac{1}{t}\sum_{k=1}^{t} \varphi(k)\varepsilon_k\right). \tag{11}$$

From the ergodicity of Y_t and the independence between $\varphi(k)$ and ε_k, we have $\frac{1}{t}\sum_{k=1}^{t} \varphi(k)\varepsilon_k \xrightarrow{a.s} E(\varphi(k)\varepsilon_k) = 0$, then $i)$ is verified.

For $ii)$, we have from (11)

$$\sqrt{t}\left(\widehat{\theta}_t - \theta\right) = \left(\frac{1}{t}\sum_{k=1}^{t} \varphi(k)\varphi(k)'\right)^{-1} \left(\frac{1}{\sqrt{t}}\sum_{k=1}^{t} \varphi(k)\varepsilon_k\right),$$

the central limit theorem for martingale differences gives

$$\frac{1}{\sqrt{t}}\sum_{k=1}^{t} \varphi(k)\varepsilon_k \xrightarrow{d} N\left(0, \sigma^2\Gamma\right),$$

leading to the result by applying Slutzky's theorem.

4. Simulation study

We simulate four restricted $EXPAR(1)$ models with $\gamma = 1$ and $\sigma = 0.1$ for the first three models, and $\sigma = 1$ for the fourth model. The selection of parameters is based on ensuring the stationarity of the models. The parameters are as follows:

*Model 1: $\theta = (-0.9, -2)'$

*Model 2: $\theta = (0.7, 1.5)'$

*Model 3: $\theta = (0.8, 1)'$

*Model 4: $\theta = (0.4, -0.9)'$.

We have added these additional models to provide a broader understanding of the estimator's performance under different parameter settings. The program has been written in R, and we have used the RLS function in the MTS package. Figures 1-8 show the behavior of estimator means and variances for models 1 to 4. We can see clearly that the parameters are very well estimated by the RLS algorithm and are consistent, the means converge to the true parameters, and the variances converge towards zero.

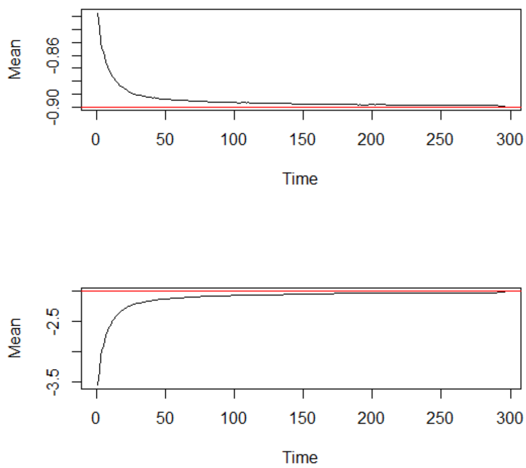

Figure 1: Mean of $\widehat{\theta}_t$ for model 1.

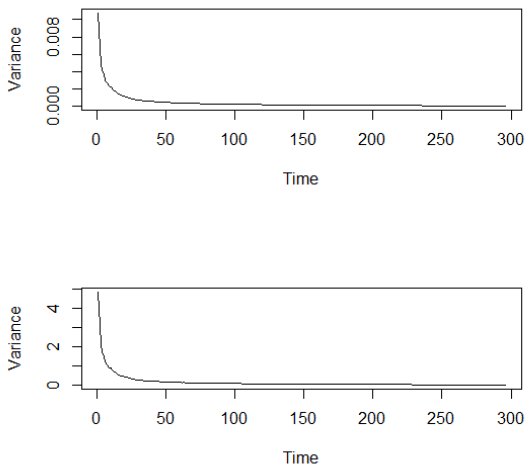

Figure 2: Variance of $\widehat{\theta}_t$ for model 1.

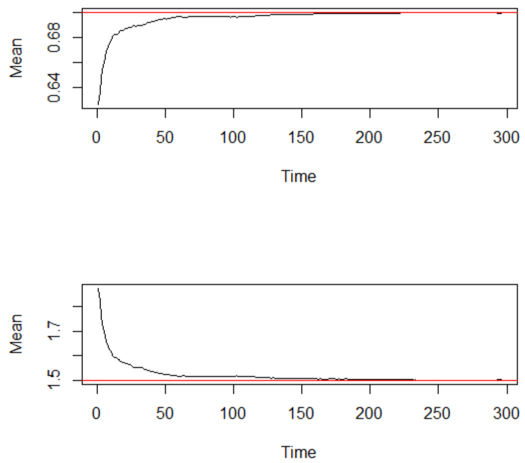

Figure 3: Mean of $\widehat{\theta}_t$ for model 2.

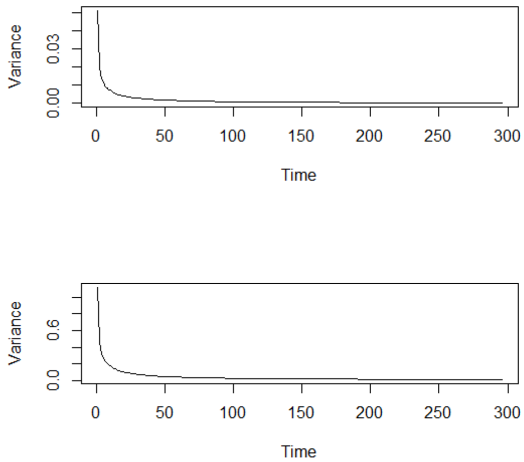

Figure 4: Variance of $\widehat{\theta}_t$ for model 2.

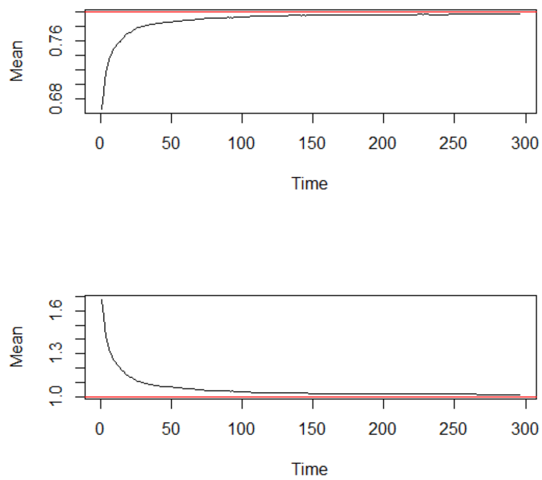

Figure 5: Mean of $\widehat{\theta}_t$ for model 3.

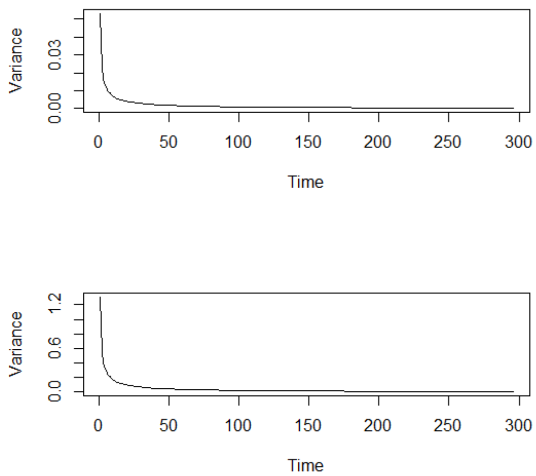

Figure 6: Variance of $\widehat{\theta}_t$ for model 3.

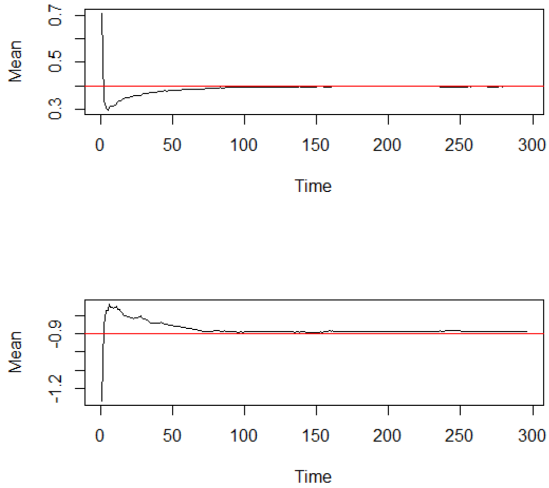

Figure 7: Mean of $\widehat{\theta}_t$ for model 4.

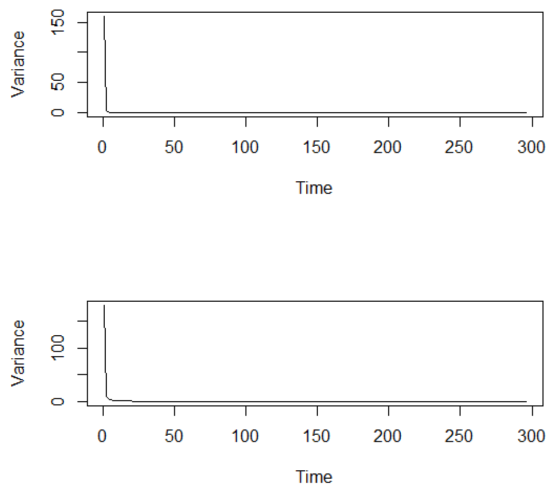

Figure 8: Variance of $\widehat{\theta}_t$ for model 4.

5. Conclusion

In this study, the RLS estimators have been proposed for the restricted $EXPAR$ model. The algorithm will be suitable in the case where the nonlinear parameter of the model is known (from previous studies) or fastly estimated in the case of real time data. The asymptotic properties of the online estimators were established and a short simulation proved their high performance.

Acknowledgements

We would like to express our sincere gratitude to the reviewers for their insightful comments and valuable suggestions, which have significantly contributed to the improvement of this paper.

References

Ljung, L. and Söderström, T. (1983). *Theory and Practice of Recursive Identification.* Cambridge, MA: MIT Press.

Messaoud, A., Weihs, C. and Hering, F. (2006). Nonlinear time series modelling: Monitoring a drilling process. In M. Spiliopoulou, R. Kruse, C. Borgelt, A. Nürnberger, & W. Gaul (Eds.), *From Data and Information Analysis to Knowledge Engineering* (pp. 302-309). Heidelberg: Springer.

Ozaki, T. (1980). Non-linear time series models for non-linear random vibrations. *Journal of Applied Probability*, 17(1), 84-93.

Shi, Z. and Aoyama, H. (1997). Estimation of exponential autoregressive time series model by using genetic algorithm. *Journal of Sound and Vibration*, 205(3), 309-321.

Shi, Z., Tamura, Y. and Ozaki, T. (2001). Monitoring the stability of BWR oscillation by nonlinear time series modeling. *Annals of Nuclear Energy*, 28(10), 953-966.

Xu, H., Ding, F. and Sheng, J. (2019). On some parameter estimation algorithms for the nonlinear exponential autoregressive model. *International Journal of Adaptive Control and Signal Processing*, 33(10). https://doi.org/10.1002/acs.3005

Xu, H., Ding, F. and Yang, E. (2020). Three-stage multi-innovation parameter estimation for an exponential autoregressive time-series model with moving average noise by using the data filtering technique. *International Journal of Robust and Nonlinear Control.* https://doi.org/10.1002/rnc.5065